Hyper-Dimensional Deformation Simulation (Supplemental Materials)

ALVIN SHI, HAOMIAO WU, and THEODORE KIM, Yale University, USA

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1 I_3 EIGENSYSTEM IN 4D

The first six eigenvalues of I_3 invariant in 4D are:

$$\lambda_0^{I_3} = \sigma_0\sigma_1 \qquad \lambda_1^{I_3} = \sigma_0\sigma_2 \qquad \lambda_2^{I_3} = \sigma_0\sigma_3 \tag{1}$$

$$\lambda_3^{I_3} = \sigma_1\sigma_2 \qquad \lambda_4^{I_3} = \sigma_1\sigma_3 \qquad \lambda_5^{I_3} = \sigma_2\sigma_3 \tag{2}$$

The corresponding generalized twist modes are:

$$\mathbf{Q}_0 = \mathbf{U}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\mathbf{V}^\top \qquad \mathbf{Q}_1 = \mathbf{U}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}\mathbf{V}^\top \tag{3}$$

$$\mathbf{Q}_2 = \mathbf{U}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\mathbf{V}^\top \qquad \mathbf{Q}_3 = \mathbf{U}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}\mathbf{V}^\top \tag{4}$$

$$\mathbf{Q}_4 = \mathbf{U}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\mathbf{V}^\top \qquad \mathbf{Q}_5 = \mathbf{U}\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\mathbf{V}^\top \tag{5}$$

The next six are generalized flip modes whose eigenvalues are negated versions of the first six. Their eigenvectors are:

$$\mathbf{Q}_6 = \mathbf{U}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\mathbf{V}^\top \qquad \mathbf{Q}_7 = \mathbf{U}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\mathbf{V}^\top \tag{6}$$

$$\mathbf{Q}_8 = \mathbf{U}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\mathbf{V}^\top \qquad \mathbf{Q}_9 = \mathbf{U}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\mathbf{V}^\top \tag{7}$$

$$\mathbf{Q}_{10} = \mathbf{U}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\mathbf{V}^\top \qquad \mathbf{Q}_{11} = \mathbf{U}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\mathbf{V}^\top \tag{8}$$

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The last four eigenvectors are linear combinations of the following stretch modes:

$$\mathbf{Q}'_0 = \mathbf{U}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\mathbf{V}^\top \qquad \mathbf{Q}'_1 = \mathbf{U}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\mathbf{V}^\top \tag{9}$$

$$\mathbf{Q}'_2 = \mathbf{U}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\mathbf{V}^\top \qquad \mathbf{Q}'_3 = \mathbf{U}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\mathbf{V}^\top \tag{10}$$

2 GENERAL HYPER-DIMENSIONAL ENERGIES

As stated in the main paper, higher order invariants appear with each additional dimension. However, for energies formulated using only the invariants I_1, I_2, and I_3, we can still derive a general eigenanalysis in any number of dimensions. Using the chain rule, the energy Hessian is the following:

$$\frac{\partial^2 \Psi(I_1, I_2, I_3)}{\partial \mathbf{F}^2} = \sum_{i=1}^{3} \frac{\partial \Psi}{\partial I_i}\mathbf{H}_i + \sum_{j,k \in \{1,2,3\}} \frac{\partial^2 \Psi}{\partial I_j \partial I_k}\mathbf{g}_j\mathbf{g}_k^\top \tag{11}$$

Eigensystems of sums are usually unpredictable. Fortunately, all the generalized twist and flip eigenvectors are eigenvectors of the energy Hessian.

To see this, recall our analyses of the individual invariant Hessians: $\mathbf{H}_2 = 2\mathbf{I}$, and the set of eigenvectors of $\mathbf{H}_1$ will always be a subset of the eigenvectors of $\mathbf{H}_3$, provided the pattern for $\mathbf{H}_3$'s eigenvectors holds generally.

For the outer products $\mathbf{g}_j\mathbf{g}_k^\top$, the general twist and flip eigenvectors are in their null space, since all $\mathbf{g}_i$ are orthogonal to all $\mathbf{q}_j$ in any dimension.

To show this, recall that $\text{Vec}(\mathbf{M})^\top\text{Vec}(\mathbf{N}) = \text{Tr}(\mathbf{M}^\top\mathbf{N})$. Furthermore, the eigenvectors $\mathbf{q}_i = \text{Vec}(\mathbf{Q}_i) = \text{Vec}(\mathbf{U}\Theta_i\mathbf{V}^\top)$, where $\mathbf{F} = \mathbf{U}\Sigma\mathbf{V}^\top$ is the deformation gradient. We note that Σ is diagonal and Θ_i always has entries of 0 in its diagonal:

$$\mathbf{g}_1^\top\mathbf{q}_i = \text{Tr}\left(\mathbf{R}^\top\mathbf{Q}_i\right) = \text{Tr}(\mathbf{V}\mathbf{U}^\top\mathbf{U}\Theta_i\mathbf{V}^\top) = \text{Tr}(\Theta_i) = 0$$
$$\mathbf{g}_2^\top\mathbf{q}_i = \text{Tr}\left(2\mathbf{F}^\top\mathbf{Q}_i\right) = 2\text{Tr}(\mathbf{V}\Sigma\mathbf{U}^\top\mathbf{U}\Theta_i\mathbf{V}^\top) = 2\text{Tr}(\Sigma\Theta_i) = 0$$
$$\mathbf{g}_3^\top\mathbf{q}_i = \text{Tr}\left(\text{adj}(\mathbf{F})^\top\mathbf{Q}_i\right) = \text{Tr}(\mathbf{V}\Sigma^{-1}\mathbf{U}^\top\mathbf{U}\Theta_i\mathbf{V}^\top) = \text{Tr}(\Sigma^{-1}\Theta_i) = 0$$

In conclusion, for the class of nD energies expressible as functions of I_1, I_2, and I_3, there are $n^2 - n$ general flip and twist eigenvectors. The remaining n eigenvectors are linear combinations of the general stretch modes, which can be found either numerically or through analytic techniques when $n < 5$.

2.1 Other Energies

2.1.1 ARAP is in The Class. The formula for ARAP is

$$\psi_{\text{ARAP}}(\mathbf{F}) = \mu\,\|\mathbf{F} - \mathbf{R}\|^2 = \mu(I_2 - 2I_1 + n) \tag{12}$$

Table 1. 4D scene timing breakdowns. Total time per frame is in **minutes:seconds** unless indicated otherwise. For most scenes, collision detections dominated, with system assembly coming in second. In the case of the noodle scene, the bowl was a kinematic collider, additionally contributing towards system assembly time. All other scenes had no kinematic obstacles. Remaining percentages were spent on slicing onto the visible hyperplane and I/O for saving geometry.

Scene	Time/Frame	Cores	Collision Detection	System Assembly	System Solve
Rotating Bunny	6.27s	16	53%	40%	4%
Armadillo Hugs	24.5s	12	49%	47%	3%
Alien Twisting	45.1s	12	57%	38%	4%
Octopus Twisting	01:25	12	65%	32%	3%
Thin Wringing	46.9s	12	58%	40%	2%
Thick Wringing	53.8s	12	53%	41%	6%
Cantilevers	14-15s	12	44%	53%	2%
4D Noodles	49.1s	12	26%	73%	1%

where $n = \mathrm{Tr}(\mathbf{R}^\top \mathbf{R})$ is the number of dimensions. Taking the Hessian:

$$\frac{\partial^2 \psi_{\mathrm{ARAP}}}{\partial \mathbf{F}^2} = 2\mu(\mathbf{I} - \mathbf{H}_1) \tag{13}$$

Applying our analysis, the eigenvectors are the generalized twists, flips, and scaling modes. The twist eigenpairs are as follows:

$$\lambda_{ij}^{\mathrm{ARAP}} = 2\mu\left(1 - \frac{2}{\sigma_i + \sigma_j}\right) \qquad \mathbf{q}_{ij}^{\mathrm{ARAP}} = \frac{1}{\sqrt{2}}\mathrm{Vec}(\mathbf{U}\Theta\mathbf{V}^\top) \tag{14}$$

Where λ^{I_1} and Θ are defined in §4.4 of the main paper. The remaining eigenvalue is 2μ, pairing off with the remaining $(n^2+n)/2$ flip and scale eigenvectors.

2.1.2 StVK is not in The Class. The formula for StVK is

$$\psi_{\mathrm{StVK}} = \mu\left\|\frac{1}{2}\left(\mathbf{F}^\top\mathbf{F} - \mathbf{I}\right)\right\|^2 + \frac{\lambda}{2}\left(\mathrm{Tr}\left(\frac{1}{2}\left(\mathbf{F}^\top\mathbf{F} - \mathbf{I}\right)\right)\right)^2 \tag{15}$$

As Kim and Eberle [2022] note, the higher order term $\mathrm{Tr}(\Sigma^4)$ is present in the energy expression. Converting it to deformation invariants actually increases in complexity with the number of dimensions. In the case of 4D, we found the following conversion:

$$\mathrm{Tr}(\Sigma^4) = \frac{1}{6}I_1^4 - 4I_3 + \frac{1}{2}I_2^2 - I_1^2 I_2 + \frac{4}{3}I_1\mathrm{Tr}(\Sigma^3) \tag{16}$$

which necessitates an analysis the invariant $\mathrm{Tr}\left(\Sigma^3\right)$. Finding a procedure for these conversions in any number of dimensions, along with analyzing the accompanying general invariants, is a possible avenue for future work.

2.2 Energy Comparisons

To compare how different deformation energies affect general physical behavior, we performed hypernoodle drops with SNH, ARAP, and StVK (Fig. 1). Qualitatively, using ARAP and SNH result in similar behaviors, while the noodles seem to "deflate" with StVK, consistent with observations of how the higher-order nature of the energy introduces a spurious minimum, resulting in undesirable behavior under compression [Kim and Eberle 2022]. The supplemental video also shows full animations.

REFERENCES

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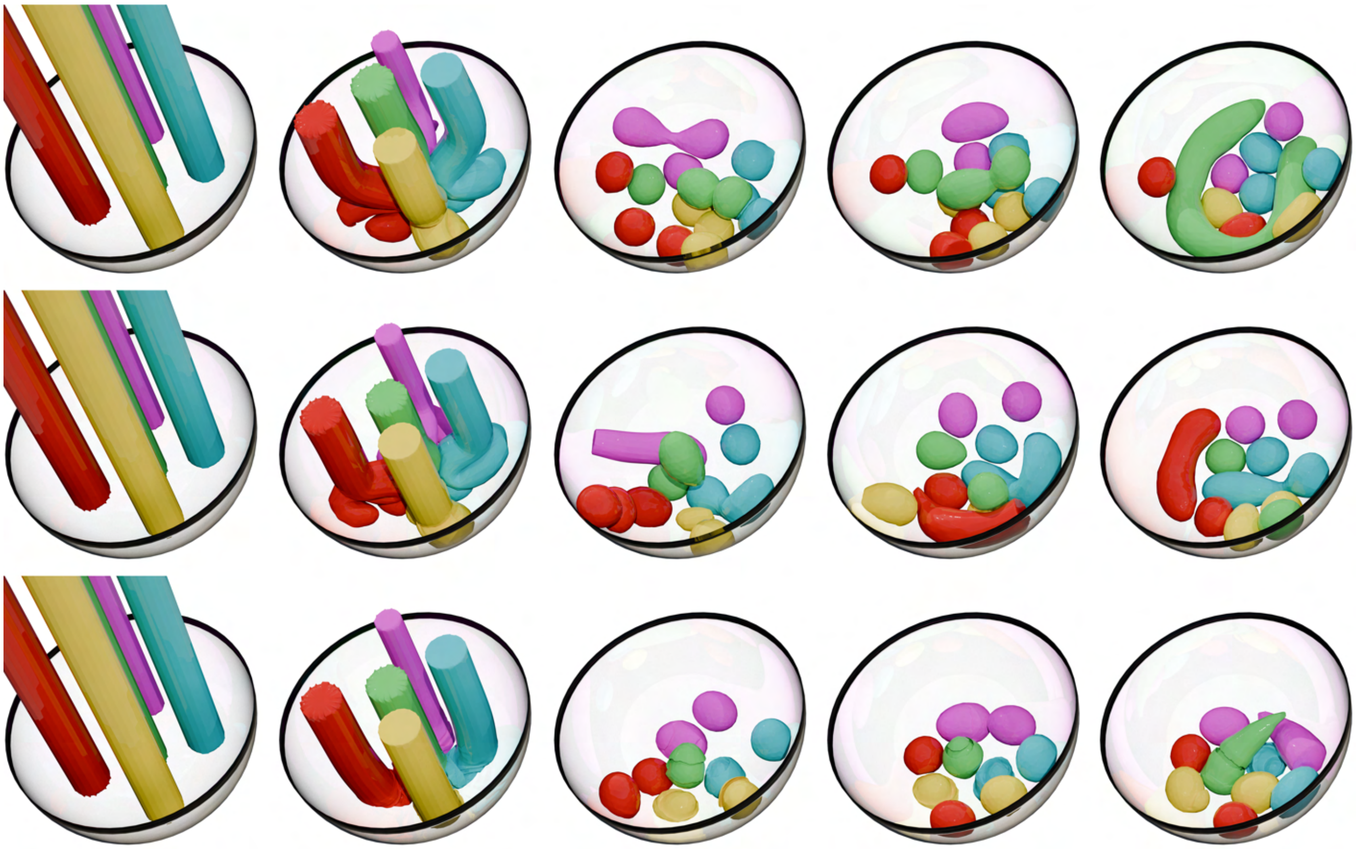

Fig. 1. *Energy Comparison*. Comparing hypernoodle drop simulations using SNH (top), ARAP (middle), and StVK (bottom) deformation energies. The SNH and ARAP are similar, while StVK exhibits inversion artifacts.